

Quadratic s -Form Field Actions with Semi-bounded Energy.

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ABSTRACT

We give in this paper a partial classification of the consistent quadratic gauge actions that can be written in terms of s -form fields. This provides a starting point to study the uniqueness of the Yang-Mills action as a deformation of Maxwell-like theories. We also show that it is impossible to write kinetic 1-form terms that can be consistently added to other 1-form actions such as tetrad gravity in four space-time dimensions even in the presence of a Minkowskian metric background.

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I Introduction

During the last years a great deal of effort has been devoted to the problem of describing consistent quantum field theories obtained by deformations of well known free Lagrangians such as the Maxwell action for electromagnetism and generalizations to p -forms [1]-[3], the Fierz-Pauli [4] model for spin 2 fields in a Minkowskian metric background and many others. In all these cases the starting point is the same: take several copies of a free action such as the ones quoted above and study the possible interaction terms that can be consistently added to them. Consistency in this context means that the number of physical degrees of freedom described by the free and deformed actions are the same and the gauge symmetries and the algebra of gauge transformations reduce to the free ones when taking the coupling constants to zero. The main results described in the previous papers are the uniqueness of the Yang-Mills action and the fact that consistent gravitational theories involving several metrics reduce to the addition of non-interacting copies of general relativity. In our opinion, there is a point that needs to be clarified: to which extent the free actions taken there as starting points are the most general ones. It is a somewhat surprising fact to realize that a complete characterization of free actions is not available. The purpose of this paper is to study a rather general class of them as a step towards its complete classification³.

In the previous examples the free actions considered are physically consistent because they have a semi-bounded energy and, hence, a well defined vacuum. This feature is kept after the introduction of interaction terms, at least for small values of the coupling constant. This leads us to demand, as a first requirement, that the free actions that we study here must satisfy that the energy be semi-bounded⁴. A first choice that we must make is the type of fields that we want to work with. As shown in [5] if one demands diff-invariance of a quadratic action one is forced to consider

³Some results in this direction, concerning quadratic diff-invariant Lagrangians, have already been published [5].

⁴An unbounded energy at the quadratic level is found in the Higgs model due to the negative mass parameter, however this is compensated by the higher degree terms in the quartic Higgs potential. We cannot rely on this kind of mechanism in the kinetic case because we do not want to have terms involving more than two derivatives.

s -form fields and the only available derivative operator is the exterior differential. One can give up diff-invariance by introducing a metric background. In this case the adjoint exterior differential (built with the help of the Hodge dual defined by the background metric) is also available. Of course, other types of geometrical objects can be considered –general tensor fields of arbitrary valence– but, to stay within the framework of Yang-Mills theories, this paper will deal only with 1-forms (although we will consider generalizations to s -form fields). Notice, however, that even in this restrictive setting some questions may be posed –and answered– concerning other types of theories, such as gravity, that can also be written purely in terms of 1-forms.

According to the discussion above we consider in this paper the most general quadratic local action that can be written in terms of 1-forms, the exterior differential and its dual in a Minkowskian space-time. We will avoid the use of mass parameters as they are expected to lead to ill behaved propagators for large momenta. As the type of analysis presented in the paper can be extended to s -forms without extra effort we will consider in section II an action depending on an arbitrary number of s -form fields with two constant arbitrary matrices P and Q that partially generalizes the action for 1-forms. In order to extract its physical content and describe its gauge symmetries one may rely on the standard Hamiltonian analysis. This, however, proves to be a cumbersome way to attack the problem because the secondary constraints that appear at several stages depend on the algebraic properties of P and Q in a non-trivial way. A superior strategy, as shown in [5], is that of using covariant symplectic techniques [6]-[9]. They are based on the direct study of the space of solutions to the field equations and the symplectic structure defined in it. The fact that we are dealing with quadratic actions (and, hence, *linear* field equations), will allow us to solve them completely and parametrize the solutions in a very convenient way. With these solutions in hand it is possible to obtain the symplectic structure that provides us with both a concrete description of the reduced phase space and the gauge symmetries present in the model. This is discussed in section III.

Once we know what the physical modes are, we want to characterize them and

select the matrices P and Q in order to have a consistent theory in the sense described before. To this end we obtain in section IV, also by using symplectic techniques, the energy-momentum and angular momentum. The main result of the paper is the partial classification of the consistent quadratic lagrangians in four dimensions that can be written in terms of s -forms. This classification is partial because some additional terms can be added for 2-form fields and we do not consider free actions with cross-terms involving different types of forms. Despite of this, the result may be useful as the starting point to study their deformations along the lines presented in [2] and set the uniqueness of Yang-Mills on a firmer footing. Another result of the paper is that it is impossible to find kinetic terms that can be consistently added to other 1-form actions such as the tetrad action for general relativity⁵

$$\int e^I \wedge e^J \wedge F_{IJ}(A) \quad , \quad (1)$$

even in the presence of a background metric such as Minkowski. This would have provided a novel way to study general relativity as an interaction term of a consistent free theory. We discuss this and give additional comments and conclusions in section V and leave computational details for the appendices.

II The Action, Field Equations, and Solutions.

We start by considering the most general quadratic, second order action that can be written in terms of 1-forms in four space-time dimensions, without the introduction of mass terms, and using the exterior differential d and its dual δ .

$$S_1[A] = \int_{\mathcal{M}} \left[dA^{\mathfrak{t}} \wedge *PdA + \delta A^{\mathfrak{t}} \wedge *Q\delta A \right] \quad . \quad (2)$$

Here \mathcal{M} is a four dimensional pseudo-Riemannian manifold⁶ without boundary with metric g that defines the Hodge dual $*$, A is a set of N 1-form fields that we write as a column vector whose transpose will be denoted by $A^{\mathfrak{t}}$; d is the exterior derivative,

⁵Here e^I is a $SO(1,3)$ valued 1-form and F_{IJ} is the curvature of a $SO(1,3)$ spin connection A_I^J .

⁶In the following we are going to work with $g_{ab} = \eta_{ab} = \text{diag}(-, +, +, +)$, the Minkowski metric in four dimensions, so we will chose $\mathcal{M} = \mathbb{R}^4$. We will denote \mathbb{R}^4 indices as a, b, \dots spatial indices as i, j, \dots and 0 will be the time index.

δ its dual and \wedge the usual exterior product (we provide a dictionary to translate between form notation and tensor notation in Appendix A). P and Q are quadratic forms represented by symmetric, real, $N \times N$ matrices. Notice that we cannot write a quadratic second order action with 1-forms only without the use of a background metric because all possible terms would be total derivatives and would cancel if \mathcal{M} has no boundary. As is well known from the study of normal modes in coupled harmonic oscillators if P or Q are positive definite they can be simultaneously diagonalized; in fact, one can find a non-singular linear redefinition of the fields that takes one of them to the identity matrix and diagonalizes the other. In this case $S_1[A]$ reduces to a sum of Maxwell actions with $(\partial_a A^a)^2$ terms that is ill defined both from the classical and the quantum point of view. If, however, P and Q are non-definite (for example singular) it may be impossible to simultaneously diagonalize them (as can be seen in simple examples), hence, one could expect that qualitatively new behaviors may occur in this case. According to this, we will allow $\text{Ker } P$ or $\text{Ker } Q$ to be different from $\{0\}$. In general we can have $\text{Ker } P \cap \text{Ker } Q \neq \{0\}$ but then we can eliminate a set of fields from (2) by a linear non-singular field redefinition. To avoid this trivial situation we demand that $\text{Ker } P \cap \text{Ker } Q = \{0\}$.

The formalism that we will use in the following is powerful enough to allow us to study generalizations of (2) for s -forms fields, so we will consider a slight modification⁷ of (2)

$$S_s[A] = \int_{\mathbb{R}^4} \left[dA^\flat \wedge *P dA + \delta A^\flat \wedge *Q \delta A \right] \quad (3)$$

where now A is a set of N s -form fields. The field equations obtained from (3) by performing variations with respect to A are

$$P \delta dA + Q d\delta A = 0 \quad . \quad (4)$$

This is a system of second order partial differential equations. The first step to solve them consists of introducing linear bases for $\text{Ker } P$, $\text{Ker } Q$ and complete them

⁷ Notice, however, that if we mix different values of s or if $s = 2$ it is possible to add other types of terms to this action. As our primary goal is the study of the 1-form case we will not consider them here.

to obtain a basis for \mathbb{R}^N . This will allow us to get a convenient set of equations from (4) that can be separately studied. We choose sets of linearly independent vectors⁸ $\{e_p\}$, $\{e_q\}$, and $\{e_r\}$ such that $\text{Ker } P = \text{Span}\{e_p\}$, $\text{Ker } Q = \text{Span}\{e_q\}$, and $\mathbb{R}^N = \text{Span}\{e_p, e_q, e_r\}$ and write $A = A^p e_p + A^q e_q + A^r e_r$. Notice that the condition $\text{Ker } P \cap \text{Ker } Q = \{0\}$ implies that $\{e_p, e_q\}$ are linearly independent. We can rewrite (4) as

$$P\delta d(A^q e_q + A^r e_r) + Qd\delta(A^p e_p + A^r e_r) = 0 \quad . \quad (5)$$

A set of necessary conditions that the solutions to (5) must satisfy can be found by acting on it with either d or δ and using $d^2 = 0$, $\delta^2 = 0$. This way we get

$$Pd\delta d(A^q e_q + A^r e_r) = 0 \quad (6)$$

$$Q\delta d\delta(A^p e_p + A^r e_r) = 0 \quad .$$

As $\{Pe_q, Pe_r\}$ are linearly independent vectors⁹ (and also $\{Qe_p, Qe_r\}$) we can write (6) as

$$d\delta dA^q = \square dA^q = 0 \quad (7)$$

$$\delta d\delta A^p = \square \delta A^p = 0$$

$$d\delta dA^r = \square dA^r = 0$$

$$\delta d\delta A^r = \square \delta A^r = 0$$

where $\square \equiv d\delta + \delta d = \partial_0^2 - \vec{\partial}^2$ is the wave operator (see Appendix A) that should not be confused with the Laplace operator—notice that we work with a Minkowskian metric—. This fact prevents us from using the Hodge decomposition for the s -forms in (7); however a suitable decomposition, that helps in solving these equations, may be found as follows. Consider

$$\delta(d\alpha - A) = 0 \quad (8)$$

⁸The letters p, q, r will be used as indices within the different subspaces defined by $\{e_p\}$, $\{e_q\}$, and $\{e_r\}$.

⁹ $\lambda^q P(e_q) + \mu^r P(e_r) = 0 \Rightarrow \lambda^q e_q + \mu^r e_r \in \text{Ker } P \Rightarrow \lambda^q e_q + \mu^r e_r + \sigma^p e_p = 0 \Rightarrow \lambda^q = \mu^r = \sigma^p = 0$.

for a given s -form A and an unknown $(s-1)$ -form α . Making the ansatz $\alpha = \delta\theta$ (equivalent in \mathbb{R}^4 to $\delta\alpha = 0$) the previous equation writes $\delta(\square\theta - A) = 0$ so by choosing θ as a solution to the inhomogeneous wave equation $\square\theta = A$ (that can always be solved under reasonable regularity conditions) we can find *some* α satisfying (8). As (8) implies the existence of a $(s+1)$ -form β such that $d\alpha - A = -\delta\beta$ we see that given A it is always possible to find forms α and β such that

$$A = d\alpha + \delta\beta \quad . \quad (9)$$

Now we are ready to solve the equations appearing in (7).

Solutions to $\square dA^q = 0$ for a s -form A^q .

Introducing (9) in $\square dA^q = 0$ we find $\square d\delta\beta^q = 0 \Leftrightarrow d\square\delta\beta^q = 0$. In \mathbb{R}^4 the last equation implies $\square\delta\beta^q = d\sigma^q$ for some σ^q . This is an inhomogeneous wave equation for $\delta\beta^q$ that gives $\delta\beta^q = \gamma^q + d\tau^q$ with $\square\gamma^q = 0$ and $\square\tau^q = \sigma^q$, so finally

$$A^q = \gamma^q + d(\alpha^q + \tau^q) \equiv \gamma^q + d\Lambda^q \quad (10)$$

where Λ^q can be taken to be arbitrary as α^q itself may be arbitrarily chosen because A^q enters the equation $\square dA^q = 0$ through the combination dA^q .

Solutions to $\square\delta A^p = 0$ for a s -form A^p .

Introducing (9) in $\square\delta A^p = 0$ we find $\square\delta d\alpha^p = 0 \Leftrightarrow \delta\square d\alpha^p = 0$. In \mathbb{R}^4 this last equation implies $\square d\alpha^p = \delta\sigma^p$ for some σ^p . This inhomogeneous wave equation for $d\alpha^p$ gives $d\alpha^p = \gamma^p + \delta\tau^p$ with $\square\gamma^p = 0$ and $\square\tau^p = \sigma^p$, so we conclude

$$A^p = \gamma^p + \delta(\tau^p + \beta^p) \equiv \gamma^p + \delta\Theta^p \quad (11)$$

where, as before, we can take Θ^p arbitrary because β^p can be chosen to be arbitrary too, as suggested by the equation $\square\delta A^p = 0$.

Solutions to $\square dA^r = 0$ and $\square\delta A^r = 0$.

A^r satisfies the equation discussed in the first place so we can always parametrize it as $A^r = \gamma^r + d\mu^r$ for some arbitrary μ^r —at this stage— and γ^r satisfying $\square\gamma^r = 0$.

Plugging this into the second equation $\square\delta A^r = 0$ gives the following fourth order equation for μ^r

$$\square\delta d\mu^r = 0 \quad . \quad (12)$$

By using the decomposition (9) for μ^r we write it as $\mu^r = \delta\varepsilon^r + d\theta^r$ and, hence, (12) gives $\square\delta d\delta\varepsilon^r = 0 \Rightarrow \delta\square^2\varepsilon^r = 0$ whose solutions have the form $\varepsilon^r = \gamma_{HD}^r + \delta\beta^r$ (with $\square^2\gamma_{HD}^r = 0$). We conclude that $\mu^r = \delta\gamma_{HD}^r + d\theta^r$ and then

$$A^r = \gamma^r + d\delta\gamma_{HD}^r \quad . \quad (13)$$

Notice that we have no arbitrariness in A^r .

As (7) are only necessary conditions we know that every solution to the field equations (4,5) can be parametrized with the help of (10), (11), and (13) but (5) imposes some further restrictions on γ^q , Λ^q , γ^p , Θ^p , γ^r , γ_{HD}^r given by

$$P\delta d(\gamma^q e_q + \gamma^r e_r) + Qd\delta(\gamma^p e_p + \gamma^r e_r + d\delta\gamma_{HD}^r e_r) = 0 \quad . \quad (14)$$

Though this last equation may look as complicated as the original field equations it is, in a sense that we make precise below, a simple *algebraic* equation that can be easily handled. Anyway some simplifications are already evident because the arbitrary objects Λ^q and Θ^p do not appear in it. In order to proceed further we need to parametrize γ^q , γ^p , γ^r , and γ_{HD}^r . To this end we take an inertial coordinate system (\vec{x}, t) in \mathbb{R}^4 and define spatial Fourier transforms as

$$f(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{w} f(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}} \quad ; \quad \frac{1}{w} f(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3\vec{x} f(\vec{x}, t) e^{-i\vec{k}\cdot\vec{x}} \quad (15)$$

with¹⁰ $w = +\sqrt{\vec{k}\cdot\vec{k}}$ introduced in the definition (15) in order to have an explicit Lorentz covariant measure. We purposely use the same letter to represent a field and

¹⁰We use 3-dimensional vector notation in \mathbb{R}^3 and denote the usual Euclidean scalar product with a dot.

its Fourier transform. We will distinguish them by their arguments. As we will be dealing with real fields we must have $f(\vec{k}, t) = \bar{f}(-\vec{k}, t)$ where in the following the bar denotes complex conjugation. We also need to perform a $3 + 1$ decomposition of the various s -forms to obtain the time and space components. For a s -form ω with components $\omega_{a_1 \dots a_s}$ we only need to consider $\omega_{i_1 \dots i_s}$ and $\omega_{0i_1 \dots i_{s-1}}$ with Fourier transforms given by

$$\begin{aligned}\omega_{i_1 \dots i_s}(\vec{k}, t) &= ik_{[i_1} \alpha_{i_2 \dots i_s]}(\vec{k}, t) + \beta_{i_1 \dots i_s}(\vec{k}, t) \\ \omega_{0i_1 \dots i_{s-1}}(\vec{k}, t) &= ik_{[i_1} a_{i_2 \dots i_{s-1}]}(\vec{k}, t) + b_{i_1 \dots i_{s-1}}(\vec{k}, t)\end{aligned}\tag{16}$$

and $\alpha_{i_1 \dots i_{s-1}}(\vec{k}, t)$, $\beta_{i_1 \dots i_s}(\vec{k}, t)$, $a_{i_1 \dots i_{s-2}}(\vec{k}, t)$, $b_{i_1 \dots i_{s-1}}(\vec{k}, t)$ satisfying the transversality conditions $k_{i_1} \alpha_{i_1 \dots i_{s-1}}(\vec{k}, t) = 0, \dots$. In the following we will give the components of a certain form (such as the previous one) by enclosing them between curly brackets as follows

$$\omega = \left\{ \begin{array}{l} ik_{[i_1} \alpha_{i_2 \dots i_s]}(\vec{k}, t) + \beta_{i_1 \dots i_s}(\vec{k}, t) \\ ik_{[i_1} a_{i_2 \dots i_{s-1}]}(\vec{k}, t) + b_{i_1 \dots i_{s-1}}(\vec{k}, t) \end{array} \right\}.$$

Solutions to the wave equation $\square \gamma = 0$ can be parametrized as in (16) with

$$\begin{aligned}w\alpha_{i_1 \dots i_{s-1}}(\vec{k}, t) &= \alpha_{i_1 \dots i_{s-1}}(\vec{k})e^{-iwt} + \bar{\alpha}_{i_1 \dots i_{s-1}}(-\vec{k})e^{iwt} \\ \beta_{i_1 \dots i_s}(\vec{k}, t) &= \beta_{i_1 \dots i_s}(\vec{k})e^{-iwt} + \bar{\beta}_{i_1 \dots i_s}(-\vec{k})e^{iwt} \\ wa_{i_1 \dots i_{s-2}}(\vec{k}, t) &= a_{i_1 \dots i_{s-2}}(\vec{k})e^{-iwt} + \bar{a}_{i_1 \dots i_{s-2}}(-\vec{k})e^{iwt} \\ b_{i_1 \dots i_{s-1}}(\vec{k}, t) &= b_{i_1 \dots i_{s-1}}(\vec{k})e^{-iwt} + \bar{b}_{i_1 \dots i_{s-1}}(-\vec{k})e^{iwt},\end{aligned}$$

where $\alpha_{i_1 \dots i_{s-1}}(\vec{k})$, $\beta_{i_1 \dots i_s}(\vec{k})$, $a_{i_1 \dots i_{s-2}}(\vec{k})$, $b_{i_1 \dots i_{s-1}}(\vec{k})$ are arbitrary transversal functions of \vec{k} only. If we consider now the solutions for A^q given by (10) we see that the arbitrariness in Λ^q allows us to absorb a piece of γ^q in Λ^q (precisely that piece of γ^q that can be written as the exterior differential of something), and hence we can write

γ^q as

$$\gamma^q = \left\{ \begin{array}{l} \beta_{i_1 \dots i_s}^q(\vec{k})e^{-iwt} + \bar{\beta}_{i_1 \dots i_s}^q(-\vec{k})e^{iwt} \\ b_{i_1 \dots i_{s-1}}^q(\vec{k})e^{-iwt} + \bar{b}_{i_1 \dots i_{s-1}}^q(-\vec{k})e^{iwt} \end{array} \right\} . \quad (17)$$

For A^p we can absorb the piece of γ^p that can be written as the adjoint exterior derivative of something in the arbitrary $\delta\Theta^p$ term and thus

$$\gamma^p = \left\{ \begin{array}{l} \frac{i}{w}k_{[i_1} \left[\alpha_{i_1 \dots i_{s-1}}^p(\vec{k})e^{-iwt} + \bar{\alpha}_{i_1 \dots i_{s-1}}^p(-\vec{k})e^{iwt} \right] \\ \frac{i}{w}k_{[i_1} \left[a_{i_1 \dots i_{s-2}}^p(\vec{k})e^{-iwt} + \bar{a}_{i_1 \dots i_{s-2}}^p(-\vec{k})e^{iwt} \right] \end{array} \right\} .$$

In the case of A^r there are no arbitrary functions, however, there is still some freedom to “move” pieces of γ^r to γ_{HD}^r because every solution to $\square\gamma^r = 0$ satisfies $\square^2\gamma^r = 0$ and γ_{HD}^r appears through $d\delta\gamma_{HD}^r$. We can, in fact write

$$\gamma^r = \left\{ \begin{array}{l} \beta_{i_1 \dots i_s}^r(\vec{k})e^{-iwt} + \bar{\beta}_{i_1 \dots i_s}^r(-\vec{k})e^{iwt} \\ \frac{i}{w}k_{[i_1} \left[a_{i_1 \dots i_{s-2}}^r(\vec{k})e^{-iwt} + \bar{a}_{i_1 \dots i_{s-2}}^r(-\vec{k})e^{iwt} \right] \end{array} \right\} \quad (18)$$

and

$$\gamma_{HD}^r = \left\{ \begin{array}{l} \frac{i}{w}k_{[i_1} \left[\alpha_{i_2 \dots i_s}^r(\vec{k}) + wt\sigma_{i_2 \dots i_s}^r(\vec{k}) \right] e^{-iwt} + \\ \quad + \frac{i}{w}k_{[i_1} \left[\bar{\alpha}_{i_2 \dots i_s}^r(-\vec{k}) + wt\bar{\sigma}_{i_2 \dots i_s}^r(-\vec{k}) \right] e^{iwt} \\ \frac{1}{s} \left[\sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) + i\alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) + iwt\sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) \right] e^{-iwt} + \\ \quad + \frac{1}{s} \left[\bar{\sigma}_{i_1 \dots i_{s-1}}^r(-\vec{k}) - i\bar{\alpha}_{i_1 \dots i_{s-1}}^r(-\vec{k}) - iwt\bar{\sigma}_{i_1 \dots i_{s-1}}^r(-\vec{k}) \right] e^{iwt} \end{array} \right\} . \quad (19)$$

The detailed derivation of (18) and (19) is given in Appendix B.

We can now plug (17)-(19) into (14) to get the final solution to the field equations. By doing this we get the conditions

$$\begin{aligned}
& P \left[isw \left(b_{i_1 \dots i_{s-1}}^q(\vec{k}) e_q + b_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right) \right] + \\
& + Q \left[w \alpha_{i_1 \dots i_{s-1}}^p(\vec{k}) e_p - isw \left(b_{i_1 \dots i_{s-1}}^r(\vec{k}) + 2\sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) \right) e_r \right] = 0, \\
\\
& P \left[w^2 \left(b_{i_1 \dots i_{s-1}}^q(\vec{k}) e_q + b_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right) \right] + \\
& + Q \left[-i w^2 s^{-1} \alpha_{i_1 \dots i_{s-1}}^p(\vec{k}) e_p - w^2 \left(b_{i_1 \dots i_{s-1}}^r(\vec{k}) + 2\sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) \right) e_r \right] = 0,
\end{aligned}$$

that give the algebraic equation

$$\begin{aligned}
P \left[b_{i_1 \dots i_{s-1}}^q(\vec{k}) e_q + b_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right] &= \\
&= Q \left[\frac{i}{s} \alpha_{i_1 \dots i_{s-1}}^p(\vec{k}) e_p + b_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r + \frac{2}{s} \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right].
\end{aligned} \tag{20}$$

This equation constraints the possible values of $b_{i_1 \dots i_{s-1}}^q(\vec{k})$, $b_{i_1 \dots i_{s-1}}^r(\vec{k})$, $\alpha_{i_1 \dots i_{s-1}}^p(\vec{k})$, $\sigma_{i_1 \dots i_{s-1}}^r(\vec{k})$; only a subset of these can be chosen as independent objects. The details of this computation can be found in Appendix C. Once we have a complete parametrization of the solutions to the field equations we must study their physical content; in particular we want to know which of the arbitrary functions describing the solution label physical degrees of freedom and which are only gauge parameters. To this end we must obtain the symplectic structure in the space of solutions.

III The Symplectic Structure: Gauge Transformations and Physical Degrees of Freedom.

We have obtained in the previous section the general solution to the field equations (4). This solution depends on a set of arbitrary, time dependent functions $\Lambda^q(\vec{k}, t)$ and $\Theta^p(\vec{k}, t)$, and on a set of arbitrary functions of \vec{k} satisfying simple algebraic constraints. This section is devoted to the computation of the symplectic structure $\Omega_{\mathcal{S}}$ in the space of solutions to the field equations \mathcal{S} . $\Omega_{\mathcal{S}}$ will provide us with several important pieces of information:

i) It will allow us to identify the physical degrees of freedom in the model and the gauge transformations¹¹.

ii) It will allow us to define the Poisson brackets in the reduced phase space; i.e. we will identify canonically conjugated pairs of variables. This is a necessary step towards the quantization of the model.

iii) The symplectic structure can be used in a very effective way to obtain conserved quantities such as the energy-momentum and angular momentum that we will use in order to impose consistency requirements on the family of models given by (3).

The action (3) defines a symplectic two-form $\Omega_{\mathcal{F}}$ in the space of fields \mathcal{F} coordinatized by $A(x)$

$$\Omega_{\mathcal{F}} = \int_{\mathbb{R}^3} J = \int_{\mathbb{R}^3} \left[dA^{\mathfrak{t}} \mathbb{A} * P dA + (Q \delta dA)^{\mathfrak{t}} \mathbb{A} * dA \right] . \quad (21)$$

We must distinguish between the ordinary exterior differential in \mathbb{R}^4 (d) and the exterior differential in \mathcal{F} ($d\mathbb{I}$). In the same way we must make a distinction between the wedge product in both cases (\wedge and \mathbb{A} respectively). We will not have to refer to any metric in \mathcal{F} so we only need a single Hodge dual symbol $*$. In all relevant cases –such as (21)– both \wedge and \mathbb{A} appear so, to make notation lighter, we will only write a \mathbb{A} symbol¹². Notice, also, that d and δ are defined in \mathbb{R}^4 but the integral in (21) is three-dimensional and, hence, we cannot “integrate by parts”. The fact that $dJ = 0$ on solutions allows us to take any space-like slice of $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. Notice that this implies that even though $A(x)$ depends on both spatial and time variables $\Omega_{\mathcal{S}}$ –the restriction of $\Omega_{\mathcal{F}}$ to $\mathcal{S} \subset \mathcal{F}$ – is time independent (we will choose inertial coordinates (\vec{x}, t)); in a sense, $\Omega_{\mathcal{S}}$ depends on equivalence classes under evolution of initial data for physical field configurations.

In order to explicitly write $\Omega_{\mathcal{S}}$ in terms of solutions we write

$$A = (\gamma^q + d\Lambda^q)e_q + (\gamma^p + \delta\Theta^p)e_p + (\gamma^r + d\delta\gamma_{HD}^r)e_r$$

¹¹It may be argued that it is possible to identify the gauge parameters directly from the solutions to the field equations due to their arbitrary time dependence; it is less obvious that the remaining \vec{k} -dependent functions are in one to one correspondence with physical degrees of freedom.

¹²This could have been avoided by explicitly writing space-time indices; we feel, however that this would unnecessarily complicate the notation.

$$\begin{aligned}
dA &= d\gamma^q e_q + (d\gamma^p + d\delta\Theta^p) e_p + d\gamma^r e_r \\
\delta A &= (\delta\gamma^q + \delta d\Lambda^q) e_q + \delta\gamma^p e_p + (\delta\gamma^r + \delta d\delta\gamma_{HD}^r) e_r \quad .
\end{aligned}$$

A tedious but straightforward computation (described in Appendix D) gives

$$\begin{aligned}
\Omega_S &= -2is! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \mathbb{d}\bar{\beta}_{i_1 \dots i_s}^{\mathbf{t}}(\vec{k}) \mathbb{A} P \mathbb{d}\beta^{i_1 \dots i_s}(\vec{k}) \\
&+ \frac{2iss!}{s-1} \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \mathbb{d}\bar{a}_{i_1 \dots i_{s-2}}^{\mathbf{t}}(\vec{k}) \mathbb{A} Q \mathbb{d}a_{i_1 \dots i_{s-2}}(\vec{k}) + \\
&+ 2iss! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \mathbb{d}\bar{b}_{i_1 \dots i_{s-1}}^r e_r^{\mathbf{t}}(\vec{k}) \mathbb{A} P \mathbb{d}b^{i_1 \dots i_{s-1}}(\vec{k}) + \\
&+ is! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \left[\mathbb{d}\bar{\sigma}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d}b^{i_1 \dots i_{s-1}}(\vec{k}) - \mathbb{d}\sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d}\bar{b}^{i_1 \dots i_{s-1}}(\vec{k}) \right] - \\
&- 2s! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \left[\mathbb{d}\bar{\alpha}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d}b^{i_1 \dots i_{s-1}}(\vec{k}) + \mathbb{d}\alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d}\bar{b}^{i_1 \dots i_{s-1}}(\vec{k}) \right]
\end{aligned} \tag{22}$$

for¹³ $0 \leq s \leq 4$ (in the case $s = 1$ the second term in (22) is absent from Ω_S and for $s = 3$ the first is zero because $\beta_{i_1 i_2 i_3}$ is transversal and hence, zero). It is important to remember that the objects appearing in (22) are subject to the algebraic constraints given by (20).

Several comments are in order now.

i) Ω_S given by (22) is *real*; although we will not write it here in terms of the real and imaginary parts of the fields.

ii) Ω_S is explicitly time independent as expected from general arguments on the time independence of the symplectic form. Notice that this comes about through rather non-trivial cancellations of terms and the explicit use of the field equations as described in the Appendix D.

iii) The functions Λ^q and Θ^p appearing in the solutions to the field equations label gauge transformations in the p and q sectors. We have not explicitly solved the algebraic constraint. Although one would have to do it in order to completely identify the physical degrees of freedom, we do not need to do it for the purpose of this paper as shown below.

¹³We use the convention that an index running from i_1 to i_0 represents the absence of an index; if the index runs from i_1 to i_{-k} for some positive integer k then the field itself is zero.

iv) When $s = 1$ the second term in (22) is absent, so it seems that no dependence in Q remains, however this is not the case because Q enters indirectly through the definition of e_r and the algebraic equation (20).

v) The gauge symmetries of the model can be traced back to the action (3) by writing

$$\begin{aligned} dA^{\mathbf{t}} \wedge *PdA &= d(A^q e_q^{\mathbf{t}} + A^r e_r^{\mathbf{t}}) \wedge *Pd(A^q e_q + A^r e_r) \\ \delta A^{\mathbf{t}} \wedge *Q\delta A &= \delta(A^p e_p^{\mathbf{t}} + A^r e_r^{\mathbf{t}}) \wedge *Q\delta(A^p e_p + A^r e_r) \quad , \end{aligned}$$

so that A^q only appears in $dA^{\mathbf{t}} \wedge *PdA$ and A^p in $\delta A^{\mathbf{t}} \wedge *Q\delta A$ and the action is invariant under

$$A^q \mapsto A^q + d\Lambda^q \quad ; \quad A^p \mapsto A^p + \delta\Theta^p \quad .$$

There is no gauge symmetry in the r sector.

Once we have $\Omega_{\mathcal{S}}$ in the physical phase space we can easily compute the energy-momentum tensor and the angular momentum. This will allow us to label physical states by their helicities and find out which conditions P and Q must satisfy in order to define a consistent theory in the sense described above.

IV Energy Momentum, Angular Momentum, and Consistency.

After finding the gauge transformations and physical degrees of freedom described by the action (3) we want to choose P and Q leading to a consistent theory. We also want to find out the helicities of the physical states (they are all, obviously, massless) to completely characterize them. In order to do this we need to obtain the energy-momentum and the angular momentum of the physical modes appearing in (22) (after solving for the algebraic constraint). This can be done in a rather convenient way by using $\Omega_{\mathcal{S}}$. This symplectic structure is invariant under all the symmetries of the theory because $\Omega_{\mathcal{F}}$ is. As is well known the degenerate directions of the symplectic form give us the gauge symmetries present in the model. Also if it

is invariant under a group of transformations and we take a vector V tangent to an orbit of this group it is straightforward to prove [6, 7], that locally $i_V \Omega_S = \mathbb{d}H$, where $i_V \Omega_S$ denotes the contraction of V and Ω_S , and the quantity H is the generator of the symmetry transformation corresponding to V . If the action is Poincaré invariant we can obtain in this way the energy-momentum and the angular momentum (with the right symmetries in their tensor indices) by computing $i_V \Omega_S$ for vectors V describing translations and Lorentz transformations and writing the result as $\mathbb{d}H$.

Space-time translations are given by

$$x^0 \equiv t \mapsto t + \tau^0 \quad ; \quad x^i \equiv \vec{x} \mapsto \vec{x} + \vec{\tau} \quad ,$$

they define a vector field in the space of solutions given by the formal expression

$$V_T = \int_{\mathbb{R}^4} d^4x \Delta_T \Phi(\vec{x}, x^0) \frac{\delta}{\delta \Phi(\vec{x}, x^0)}$$

where $\Phi(\vec{x}, x^0)$ labels the solutions to the field equations , $\frac{\delta}{\delta \Phi(\vec{x}, x^0)}$ denotes the functional derivative and $\Delta_T \Phi(\vec{x}, x^0)$ is the translation on $\Phi(\vec{x}, x^0)$ with parameters $(\vec{\tau}, \tau^0)$. The Fourier transforms of $\Delta_T \Phi(\vec{x}, x^0)$ for the fields appearing in (22) are

$$\begin{aligned} \Delta_T \beta_{i_1 \dots i_s}(\vec{k}) &= i\tau^a k_a \beta_{i_1 \dots i_s}(\vec{k}) \\ \Delta_T a_{i_1 \dots i_{s-1}}(\vec{k}) &= i\tau^a k_a a_{i_1 \dots i_{s-1}}(\vec{k}) \\ \Delta_T b_{i_1 \dots i_{s-1}}(\vec{k}) &= i\tau^a k_a b_{i_1 \dots i_{s-1}}(\vec{k}) \\ \Delta_T \sigma_{i_1 \dots i_{s-1}}(\vec{k}) &= i\tau^a k_a \sigma_{i_1 \dots i_{s-1}}(\vec{k}) \\ \Delta_T \alpha_{i_1 \dots i_{s-1}}^p(\vec{k}) &= i\tau^a k_a \alpha_{i_1 \dots i_{s-1}}^p(\vec{k}) \\ \Delta_T \alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) &= i\tau^a k_a \alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) + \tau^0 w \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) \quad , \end{aligned}$$

with $k^0 \equiv w$. We do not need the transformations for Λ^p, Θ^q . Notice that $\alpha_{i_1 \dots i_{s-1}}^r(\vec{k})$ transforms in a rather peculiar way due to the terms $w\sigma_{i_1 \dots i_{s-1}}^r(\vec{k})$ and $w\bar{\sigma}_{i_1 \dots i_{s-1}}^r(-\vec{k})$ in (19). For the fields appearing in Ω_S we have

$$\begin{aligned} i_{V_T} \mathbb{d} \beta_{i_1 \dots i_s}(\vec{k}) &= i\tau^a k_a \beta_{i_1 \dots i_s}(\vec{k}) \\ i_{V_T} \mathbb{d} \sigma_{i_1 \dots i_{s-1}}(\vec{k}) &= i\tau^a k_a \sigma_{i_1 \dots i_{s-1}}(\vec{k}) \\ i_{V_T} \mathbb{d} a_{i_1 \dots i_{s-2}}(\vec{k}) &= i\tau^a k_a a_{i_1 \dots i_{s-2}}(\vec{k}) \end{aligned}$$

$$\begin{aligned}
i_{V_T} \mathbb{D} \alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) &= i\tau^a k_a \alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) + \tau^0 w \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) \\
i_{V_T} \mathbb{D} b_{i_1 \dots i_{s-1}}(\vec{k}) &= i\tau^a k_a b_{i_1 \dots i_{s-1}}(\vec{k}) \quad ,
\end{aligned}$$

and hence the energy momentum can be obtained as

$$\begin{aligned}
i_{V_T} \Omega &= \tau^a \mathbb{D} P_a = \\
&\mathbb{D} \left\{ \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \left[-2s! (\tau^a k_a) \bar{\beta}_{i_1 \dots i_s}^{\mathbf{t}}(\vec{k}) P \beta^{i_1 \dots i_s}(\vec{k}) + \frac{2ss!}{s-1} (\tau^a k_a) \bar{a}_{i_1 \dots i_{s-2}}^{\mathbf{t}}(\vec{k}) Q a^{i_1 \dots i_{s-2}}(\vec{k}) + \right. \right. \\
&\quad + 2ss! (\tau^a k_a) \bar{b}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P b^{i_1 \dots i_{s-1}}(\vec{k}) + \\
&\quad + s! (\tau^a k_a) \bar{\sigma}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P b^{i_1 \dots i_{s-1}}(\vec{k}) + s! (\tau^a k_a) \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P \bar{b}^{i_1 \dots i_{s-1}}(\vec{k}) + \\
&\quad + 2is! (\tau^a k_a) \bar{\alpha}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P b^{i_1 \dots i_{s-1}}(\vec{k}) + 2is! (\tau^a k_a) \alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P \bar{b}^{i_1 \dots i_{s-1}}(\vec{k}) - \\
&\quad \left. \left. - s! (\tau^0 w) \bar{\sigma}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P b^{i_1 \dots i_{s-1}}(\vec{k}) - s! (\tau^0 w) \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P \bar{b}^{i_1 \dots i_{s-1}}(\vec{k}) \right] \right\}. \tag{23}
\end{aligned}$$

In view of (23) it must be pointed out that the appearance of τ^0 does not spoil the Lorentz covariance of P_a because $\alpha_{i_1 \dots i_{s-1}}^r(\vec{k})$ have transformation laws under space-time translations¹⁴ that involve τ^0 . Also notice that the second term is absent when $s = 1$ and the first one is zero when $s = 3$. We consider now spatial rotations in order to identify the helicities of the physical states described by the action (3). To this end we need

$$\begin{aligned}
i_{V_R} \mathbb{D} \beta_{i_1 \dots i_s}(\vec{k}) &= -s \varepsilon_{[i_1|jk} \Lambda_j \beta_{k|i_2 \dots i_s]}(\vec{k}) - \varepsilon_{jkl} k_j \Lambda_k \frac{\partial \beta_{i_1 \dots i_s}}{\partial k_l}(\vec{k}) \\
i_{V_R} \mathbb{D} a_{i_1 \dots i_{s-2}}(\vec{k}) &= -(s-2) \varepsilon_{[i_1|jk} \Lambda_j a_{k|i_2 \dots i_{s-2}]}(\vec{k}) - \varepsilon_{jkl} k_j \Lambda_k \frac{\partial a_{i_1 \dots i_{s-2}}}{\partial k_l}(\vec{k}) \\
i_{V_R} \mathbb{D} b_{i_1 \dots i_{s-1}}(\vec{k}) &= -(s-1) \varepsilon_{[i_1|jk} \Lambda_j b_{k|i_2 \dots i_{s-1}]}(\vec{k}) - \varepsilon_{jkl} k_j \Lambda_k \frac{\partial b_{i_1 \dots i_{s-1}}}{\partial k_l}(\vec{k}) \\
i_{V_R} \mathbb{D} \sigma_{i_1 \dots i_{s-1}}(\vec{k}) &= -(s-1) \varepsilon_{[i_1|jk} \Lambda_j \sigma_{k|i_2 \dots i_{s-1}]}(\vec{k}) - \varepsilon_{jkl} k_j \Lambda_k \frac{\partial \sigma_{i_1 \dots i_{s-1}}}{\partial k_l}(\vec{k}) \\
i_{V_R} \mathbb{D} \alpha_{i_1 \dots i_{s-1}}(\vec{k}) &= -(s-1) \varepsilon_{[i_1|jk} \Lambda_j \alpha_{k|i_2 \dots i_{s-1}]}(\vec{k}) - \varepsilon_{jkl} k_j \Lambda_k \frac{\partial \alpha_{i_1 \dots i_{s-1}}}{\partial k_l}(\vec{k})
\end{aligned}$$

¹⁴These objects have also unusual transformation laws under Lorentz boosts.

and hence

$$i_{V_R}\Omega_S = \varepsilon_{ijk}\Lambda_i \mathbb{D}J_{jk} = \quad (24)$$

$$\begin{aligned} & \mathbb{D}\left\{\varepsilon_{ijk}\Lambda_i \int_{\mathbf{R}^3} \frac{d^3\vec{k}}{w} \left[-2iss!k_j \frac{\partial \bar{\beta}_{i_1 \dots i_s}^{\mathbf{t}}(\vec{k})}{\partial k_k} P\beta_{i_1 \dots i_s}(\vec{k}) + \right. \right. \\ & + \frac{2iss!}{s-1} k_j \frac{\partial \bar{a}_{i_1 \dots i_{s-2}}^{\mathbf{t}}(\vec{k})}{\partial k_k} Qa_{i_1 \dots i_{s-2}}(\vec{k}) + 2iss!k_j \frac{\partial \bar{b}_{i_1 \dots i_{s-1}}^{\mathbf{t}}(\vec{k})}{\partial k_k} Pb_{i_1 \dots i_{s-1}}(\vec{k}) - \\ & - is!k_j \left(\frac{\partial \bar{\sigma}_{i_1 \dots i_{s-1}}^r(\vec{k})}{\partial k_k} e_r^{\mathbf{t}} Pb_{i_1 \dots i_{s-1}}(\vec{k}) - \frac{\partial \sigma_{i_1 \dots i_{s-1}}^r(\vec{k})}{\partial k_k} e_r^{\mathbf{t}} P\bar{b}_{i_1 \dots i_{s-1}}(\vec{k}) \right) + \\ & + 2s!k_j \left(\frac{\partial \bar{\alpha}_{i_1 \dots i_{s-1}}^r(\vec{k})}{\partial k_k} e_r^{\mathbf{t}} Pb_{i_1 \dots i_{s-1}}(\vec{k}) + \frac{\partial \alpha_{i_1 \dots i_{s-1}}^r(\vec{k})}{\partial k_k} e_r^{\mathbf{t}} P\bar{b}_{i_1 \dots i_{s-1}}(\vec{k}) \right) + \\ & + 2iss!\bar{\beta}_{ji_2 \dots i_s}^{\mathbf{t}}(\vec{k}) P\beta_{ki_2 \dots i_s}(\vec{k}) - \frac{2i(s-2)ss!}{s-1} \bar{a}_{ji_2 \dots i_{s-2}}^{\mathbf{t}}(\vec{k}) Qa_{ki_2 \dots i_{s-2}}(\vec{k}) - \\ & - 2i(s-1)ss!\bar{b}_{ji_2 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} Pb_{ki_2 \dots i_{s-2}}(\vec{k}) - \\ & - i(s-1)s! \left(\bar{\sigma}_{ji_2 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} Pb_{ki_2 \dots i_{s-1}}(\vec{k}) - \sigma_{ji_2 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P\bar{b}_{ki_2 \dots i_{s-1}}(\vec{k}) \right) + \\ & \left. + 2(s-1)s! \left(\bar{\alpha}_{ji_2 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} Pb_{ki_2 \dots i_{s-1}}(\vec{k}) + \alpha_{ji_2 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} P\bar{b}_{ki_2 \dots i_{s-1}}(\vec{k}) \right) \right] \Big\}, \end{aligned}$$

we will use this later in order to identify the helicities of the physical modes for choices of P and Q leading to consistent models.

In the following we will find the conditions that P and Q must satisfy in order to define a consistent theory. The main condition that we will impose is the semi-boundedness of the energy. We need this to ensure that, after coupling the fields to some others or with themselves via self-interaction terms, we have a stable theory. If we look at the energy-momentum given by (23) we see that $\beta_{i_1 \dots i_s}(\vec{k})$ and $a_{i_1 \dots i_{s-2}}(\vec{k})$ are decoupled from the remaining modes so, to have a positive definite or semi-definite energy both P and Q , when present in (23), must be definite or semi-definite. Notice that in the case $s = 1$ (1-form fields) the term involving $a_{i_1 \dots i_{s-2}}$ is absent and hence we only have a condition on P ; if $s = 2$ we have conditions both on P and Q and, finally, if $s = 3$ we only have conditions on Q . The remaining terms in (23) are proportional to e_r . From the fact that each of them is also proportional to Pb it is very easy to prove that none of the terms involving b , α , and σ can be zero for non zero values of the fields if the e_r sector is present. This is so because the projection of $\text{Im } P$ on e_r is

always non-zero because $\text{Im } P$ is orthogonal to $\text{Ker } P$ (P is a symmetric matrix) and e_r is orthogonal to $\text{Ker } P$. We see then that α , b , and σ give a non-zero contribution to the energy. However, the quadratic form that defines the energy has some zeroes in its diagonal, in particular there are no $\bar{\alpha}^r - \alpha^r$ terms. This fact is independent of the algebraic constraint because it does not involve α^r . A well known result in linear algebra (see Appendix E) states that, under the previous conditions, a quadratic form with a zero in its main diagonal can never be neither definite nor semi-definite. We conclude that¹⁵ if $e_r \neq 0$ the energy cannot be semi-bounded and hence the action (23) leads to an inconsistent theory.

We consider now the case $e_r = 0$ for which only the terms containing β and a remain. Clearly it suffices now to choose P and Q (in those cases in which the corresponding terms are present in (23)) to be definite or semi-definite. The question to answer at this point is whether these models can be non-trivial in the sense that no linear transformation of the fields –we want to preserve the quadratic character of the action– takes the action (3) to the form

$$\int_{\mathbf{R}^4} \left[\sum_{g=1}^{n_d} \sigma_g dA_g \wedge *dA_g + \sum_{g=n_d+1}^{n_d+n_\delta} \sigma_g \delta A_g \wedge *\delta A_g \right] , \quad (25)$$

with $\sigma_g = \pm 1$ (and always positive whenever the corresponding piece describes local degrees of freedom). Notice that, for example, in the case $s = 1$ (25) is the sum of several Maxwell actions and $(\delta A)^2$ terms that carry no degrees of freedom in a Minkowskian space-time.

Let us take P and Q such that¹⁶ $\mathbb{R}^N = \text{Ker } P \oplus \text{Ker } Q$ so that $e_r = 0$. By means of a linear transformation we can take P to a diagonal form with $(\dim \text{Ker } P)$ -elements equal to one and the rest equal to zero. Let us restrict us for the moment to the cases $s = 1$ or $s = 2$ and suppose then that P is positive semi-definite. We can then write

¹⁵Notice that the effect of not having a e_r sector can be taken into account by setting $e_r = 0$ in the previous formulas; in such case a basis of \mathbb{R}^N is spanned only by e_p and e_q .

¹⁶Remember that $\text{Ker } P \cap \text{Ker } Q = \{0\}$ so that the sum is indeed a direct sum of vector subspaces of \mathbb{R}^N . Also $\text{Ker } P$ and $\text{Ker } Q$ need not be mutually orthogonal.

in block form as

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \quad (26)$$

The most general linear, non singular, field redefinition leaving it invariant is given by the matrix

$$R = \begin{bmatrix} \alpha & \alpha_1 \\ 0 & \alpha_2 \end{bmatrix} \quad (27)$$

with α orthogonal and α_2 non-singular. If we write Q , with the same block dimensions of (26) as

$$Q = \begin{bmatrix} a & b \\ b^\mathbf{t} & c \end{bmatrix} ,$$

it transforms under (27) according to

$$RQR^\mathbf{t} = \begin{bmatrix} (\alpha a + \alpha_1 b^\mathbf{t})\alpha^\mathbf{t} + (\alpha b + \alpha_1 c^\mathbf{t})\alpha_1^\mathbf{t} & (\alpha b + \alpha_1 c)\alpha_2^\mathbf{t} \\ \alpha_2(\alpha b + \alpha_1 c)^\mathbf{t} & \alpha_2 c \alpha_2^\mathbf{t} \end{bmatrix} \quad (28)$$

Let us suppose now that c is non singular as a $(\dim \text{Ker } P) \times (\dim \text{Ker } P)$ matrix; then, $\alpha_1 = -\alpha b c^{-1}$ would render the non-diagonal blocks of (28) equal to zero. Furthermore, α and α_2 can be chosen in such a way that (28) is diagonal; in fact c can be diagonalized by an orthonormal α_2 . Taking into account that $\text{rank } Q = \dim \text{Ker } P = \text{rank } c$ which implies $(a - b c^{-1} b^\mathbf{t}) = 0$ we find $(\alpha a + \alpha_1 b^\mathbf{t})\alpha^\mathbf{t} = \alpha(a - b c^{-1} b^\mathbf{t})\alpha^\mathbf{t} = 0$. We see that by taking a non-singular c we can transform the action (3) into (25) by a non-singular linear field redefinition.

Let us suppose now that c is singular; in this case, one can find non-zero vectors $\rho_c \in \mathbb{R}^{\dim \text{Ker } P}$ such that $c\rho_c = 0$ so if $b\rho_c \neq 0$ there is no way to mutually diagonalize both P and Q because the off-diagonal blocks in (28) would be non-zero. If we look at $\text{Ker } Q$ we have

$$\left\{ \begin{bmatrix} a & b \\ b^\mathbf{t} & c \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} ay + bx = 0 \\ b^\mathbf{t}y + cx = 0 \end{matrix} \right\} . \quad (29)$$

The second equation in (29) implies $\rho_c^\mathbf{t} b^\mathbf{t} y = 0$ which tells us that the number of independent y vectors is strictly smaller than $N - \dim \text{Ker } P$. This means that is

impossible to have a basis for \mathbb{R}^N built only with vectors belonging to $\text{Ker } P$ and $\text{Ker } Q$; i.e. we would have some vector e_r different from zero. We conclude then that the requirement that $e_r = 0$ forces us to choose P and Q in such a way that they can be simultaneously diagonalized leading to a “trivial” action of the type (25).

Finally, notice that for a singular c such that for every $\rho_c \in \text{Ker } c$ we have $b\rho_c = 0$ we can still make the non-diagonal blocks equal to zero and from the fact that a symmetric singular matrix admits a symmetric pseudoinverse¹⁷ we can easily prove that Q and P are simultaneously diagonalizable.

The case $s = 3$ can be analyzed by following the same lines just by switching the roles of P and Q .

In view of the previous discussion we see that, whenever the energy is positive definite or semi-definite –something that happens only when $e_r = 0$ – the action can be transformed by means of a non-singular linear field redefinition into an action of the type (25). When $e_r = 0$ it is very easy to analyze the helicities of the physical modes as only the terms involving β and a should be taken into account. The coefficients of these terms are the same as those of the corresponding terms in the symplectic form. (22). For $s = 1$ we have helicities ± 1 , for $s = 2$ we have scalars (notice that for $s = 2$ and $e_r = 0$ the spin part of (24) is zero) and for $s = 3$ we find again helicities ± 1 .

Another important consequence of the previous arguments is the impossibility of adding a kinetic term written in terms of 1-forms to the tetrad gravity action (1). This is so because e_I and $A_I{}^J$ transform as an internal vector and a connection respectively. The connection can be identified as a field coming from the first term of (25) after a suitable deformation of the Yang-Mills type. However, the transformation law of e_I is such that it cannot be derived neither from a connection in the first term (P -term) of (25) nor a field in the Q -term.

It is very important to understand the key role played by the condition that the energy be definite or semi-definite in this respect. If one relaxes this condition, it is actually possible to find actions with well defined propagators after the gauge fixing

¹⁷ $Cx = y$ implies $PCP^\mathbf{t}Px = Py$ and hence $DPx = Py$ where D is diagonal. We can write $Px = D^-Py$ with D^- consisting on the inverses of the non-zero eigenvalues of C so that $x = P^\mathbf{t}D^-Py$ which proves that there always exist a symmetric pseudoinverse.

and with diff-invariant interaction terms very similar to the tetrad action (1). Let us consider for example

$$S = \int_{\mathbf{R}^4} \left[\nabla e^I \wedge * \nabla e_I + F^I \wedge * \nabla e_I + De^I \wedge * De_I + \varepsilon_{IJK} e^I \wedge e^J \wedge F^K \right] \quad , \quad (30)$$

where now $I = 1, 2, 3$ label (internal) $SO(3)$ indices, ε_{IJK} in the 3-dimensional totally antisymmetric object, $\nabla e^I = de^I + [A, e]^I$, $F^I = 2dA^I + [A, A]^I$, $De^I = \delta e^I + [i_A, e]^I$. This action has a quadratic term leading to a well defined propagator after gauge fixing, is power-counting renormalizable and has, as an interacting term, the action for the Husain-Kuchař model¹⁸ which mimics a term of the type defined by (1). It also has the property that there are no regular field redefinitions that allow us to remove the $F^I \wedge \nabla e_I$ term to make the kinetic term diagonal. It is only the fact that the energy is not semi-bounded that leads to inconsistencies. This is reminiscent, but not equal, to the well known behavior of higher derivative theories of gravity. The reader may argue that this is to be expected due to the presence of the $De_I \wedge * De^I$ term in (30) as terms similar to this are known to spoil, for example, the familiar Maxwell action. Though, at the end of the day, this happens to be the case, the reasons, as shown above, are not obvious. In fact, there are actions involving δe that are consistent (albeit trivial).

V Conclusions and Comments.

The first conclusion of the paper is that the free actions considered in the literature [1]-[3] to study consistent interactions between gauge fields are not the most general ones in a precise sense. In these papers the starting point is always a Maxwell-type of action for s -form fields. In some instances; for example 1-forms, one can somehow extend some of the results already known for 1-forms to 3-forms fields because the second term in (2) can be considered as a Maxwell action for 3-forms. If one does not allow interactions between the “ P -sector” and the “ Q -sector” the results in the literature already apply to this case. Notice, however, that the 2-form case is different

¹⁸The Husain-Kuchař model [10] is a toy model for general relativity that has 3 local degrees of freedom per space point due to the absent of a scalar constraint in its Hamiltonian formulation.

in this respect as both terms in (3) can be interpreted in terms of 2-forms as disjoint sectors of a Maxwell-like action.

Our result is useful because it gives general free actions that should be taken as starting points to the study of their consistent interactions so we hope that a deeper knowledge about the uniqueness of Yang-Mills can be achieved by considering their deformations.

We want to remark that the use of covariant symplectic techniques for quadratic theories is very convenient in several respects. First of all it is much simpler than the use of the familiar Dirac formalism. There, the appearance of successive layers of secondary constraints depending on the algebraic properties of the P and Q matrices requires tedious computations to disentangle the structure of the phase space, constraints, and gauge symmetries. Here, as also shown in [5], the possibility of explicitly solving the field equations (via Fourier transform and after a suitable $3 + 1$ splitting) allows us to use the covariant symplectic formalism to describe the physical degrees of freedom and symmetries of the model. As we have seen, it is actually easy to find the full symmetries of the Lagrangian (rather than the 3-dimensional version provided by the constraints in the Hamiltonian formalism). We have seen also that these symplectic techniques help in the derivation of the energy-momentum and angular momentum, key ingredients to study the particle content and consistency of the actions considered in this paper.

The use of these covariant symplectic techniques offers the possibility of studying in a very systematic way whole families of quadratic actions for different kinds of fields. Our point of view is that the only way to build a perturbatively consistent theory is to completely understand the quadratic part of the action and the subsequent deformations of it. We think that no systematic study of quadratic gauge actions has been carried out to date. This paper and the previous one dealing with the diff-invariant case, are first steps in the program of characterizing large sets of gauge quadratic actions. We hope that interesting theories may appear in this search hopefully leading to a new understanding of Yang-Mills and other theories such as

general relativity. For example, as shown in [11], the Husain-Kuchař model can be described by coupling two BF Lagrangians with a quadratic part involving cross-terms with 1 and 2-forms. An open and interesting question is if one can find a quadratic action with consistent deformations that include the Husain-Kuchař model in its BF description. Work in this direction is in progress.

A second question that has been answered in passing concerns the impossibility of adding quadratic 1-form terms to the tetrad action for gravity, even in the presence of a metric background such as Minkowski. This result goes beyond the negative conclusion of [5] where we showed that no diff-invariant kinetic terms could be consistently added to the gravitational action. Here we have seen that the only consistent actions in terms of 1-forms are just Maxwell actions and $(\delta A)^2$ actions (that describe no degrees of freedom in a Minkowski background), so even in the presence of a background it is impossible to find suitable kinetic terms.

As emphasized above the requirement that the energy be definite or semi-definite is crucial. It is also important to realize that one should not be tempted to believe that the presence of $\delta A^\flat \wedge *Q\delta A$ terms trivially leads to an inconsistent theory; in fact if $P = 0$ the action is consistent albeit trivial. The presence of the $\{e_r\}$ sector and its detailed structure is the key element to explain why the theory is inconsistent in many cases.

Appendix A.

As shown in the paper it is, at times, quite useful to translate from tensor notation to index-free form notation so we provide in this appendix a dictionary to go from one representation to the other. This will also allow us to fix several conventions needed when writing forms as totally covariant antisymmetric tensors.

We will write a s -form ω defined on a differentiable manifold \mathcal{M} of dimension N (endowed with coordinates x^a) as

$$\omega(x) = \omega_{a_1 \dots a_s}(x) dx^{a_1} \wedge \dots \wedge dx^{a_s}$$

with

$$\omega_{a_1 \dots a_s} = \omega_{[a_1 \dots a_s]} \equiv \frac{1}{s!} \sum_{\pi \in \mathcal{S}_s} (-1)^\pi \omega_{\pi(a_1) \dots \pi(a_s)} \cdot (\pi \in \mathcal{S}_s \text{ is a permutation of order } s).$$

The exterior (wedge) product of a s -form ω and a r -form ξ is defined as

$$\omega \wedge \xi = \omega_{[a_1 \dots a_s} \xi_{b_1 \dots b_r]} dx^{a_1} \wedge \dots \wedge dx^{a_s} \wedge dx^{b_1} \wedge \dots \wedge dx^{b_r} \quad ,$$

and satisfies

$$\begin{aligned} \omega \wedge \xi &= (-1)^{sr} \xi \wedge \omega \quad , \\ (\xi \wedge \eta) \wedge \omega &= \xi \wedge (\eta \wedge \omega) \quad . \end{aligned}$$

We define the exterior differential that takes a s -form ω to a $(s+1)$ -form as

$$d\omega = \partial_{[a_1} \omega_{a_2 \dots a_{s+1}]} dx^{a_1} \wedge \dots \wedge dx^{a_{s+1}} \quad ,$$

and satisfies

$$\begin{aligned} d^2 &= 0 \\ d(\omega \wedge \xi) &= d\omega \wedge \xi + (-1)^s \omega \wedge d\xi \quad . \end{aligned}$$

In the presence of a non-degenerate metric in \mathcal{M} we can define the Hodge dual of a s -form ω as the $(N-s)$ -form given by

$$*\omega = \frac{1}{(N-s)!} \frac{1}{\sqrt{|\det g|}} \omega_{b_1 \dots b_s} \tilde{\eta}^{b_1 \dots b_s c_1 \dots c_{N-s}} g_{a_1 c_1} \dots g_{a_{N-s} c_{N-s}} dx^{a_1} \wedge \dots \wedge dx^{a_{N-s}} \quad ,$$

where $\tilde{\eta}^{b_1 \dots b_N}$ is the Levi-Civita tensor density on \mathcal{M} defined to be, in any coordinate chart, $+1$ for even permutations of the indices and -1 for odd permutations. If g_{ab} has Riemannian signature we have

$$**\omega = (-1)^{s(N-s)} \omega$$

whereas for Lorentzian signatures we find

$$**\omega = (-1)^{s(N-s)+1} \omega \quad .$$

We can also define the adjoint exterior differential δ as

$$\begin{aligned}\delta &= (-1)^{N(s+1)+1} * d * && \text{Riemannian signature} \\ \delta &= (-1)^{N(s+1)} * d * && \text{Lorentzian signature.}\end{aligned}$$

It takes s -forms to $(s-1)$ -forms according to

$$\delta\omega = -s\nabla^a\omega_{aa_1\dots a_{s-1}}dx^{a_1}\wedge\dots\wedge d^{a_{s-1}} \quad ,$$

where ∇ is the metric compatible, torsion-free, covariant derivative and satisfies

$$\delta^2 = 0 \quad .$$

Finally we define the wave operator

$$\square = d\delta + \delta d \quad ,$$

that takes s -forms to s -forms and is given by

$$\square\omega = -\nabla_a\nabla^a\omega_{a_1\dots a_s}dx^{a_1}\wedge\dots\wedge x^{a_s};$$

it commutes with both d and δ .

As already stated in the main text we will refer to the components of differential forms in a Minkowskian background as follows

$$\omega = \left\{ \begin{array}{c} \omega_{i_1\dots i_s} \\ \omega_{0i_1\dots i_{s-1}} \end{array} \right\} \quad .$$

With the definition for the Fourier transform given by (15) we have the following useful formulas for the various differential operators acting on s -forms. Let us write a general s -form ω as

$$\omega(\vec{k}, t) = \left\{ \begin{array}{c} ik_{[i_1}\alpha_{i_2\dots i_s]}(\vec{k}, t) + \beta_{i_1\dots i_s}(\vec{k}, t) \\ ik_{[i_1}a_{i_2\dots i_{s-2}}](\vec{k}, t) + b_{i_1\dots i_{s-1}}(\vec{k}, t) \end{array} \right\}$$

with $\alpha_{i_1 \dots i_{s-1}}$, $\beta_{i_1 \dots i_s}$, $a_{i_1 \dots i_{s-2}}$, and $b_{i_1 \dots i_{s-1}}$ transversal. We have now (dots represent time derivatives and $w = +\sqrt{\vec{k} \cdot \vec{k}}$)

$$\begin{aligned}
d\omega &= \left\{ \begin{array}{c} ik_{[i_1} \beta_{i_2 \dots i_{s+1}]} \\ ik_{[i_1} \left(\frac{1}{s+1} \dot{\alpha}_{i_2 \dots i_s]} - \frac{s}{s+1} b_{i_2 \dots i_s]} \right) + \frac{1}{s+1} \dot{\beta}_{i_1 \dots i_s} \end{array} \right\} \\
\delta\omega &= \left\{ \begin{array}{c} ik_{[i_1} s \dot{a}_{i_2 \dots i_{s-1}]} + w^2 \alpha_{i_1 \dots i_{s-1}} + s \dot{b}_{i_1 \dots i_{s-1}} \\ -\frac{s}{s-1} w^2 a_{i_1 \dots i_{s-2}} \end{array} \right\} \\
d\delta\omega &= \left\{ \begin{array}{c} ik_{[i_1} \left(s \dot{b}_{i_2 \dots i_s]} + w^2 \alpha_{i_2 \dots i_s]} \right) \\ ik_{[i_1} \left(\ddot{a}_{i_2 \dots i_{s-1}]} + w^2 a_{i_2 \dots i_{s-1}]} \right) + \ddot{b}_{i_1 \dots i_{s-1}} + \frac{w^2}{s} \dot{\alpha}_{i_1 \dots i_{s-1}} \end{array} \right\} \\
\delta d\omega &= \left\{ \begin{array}{c} ik_{[i_1} \left(\ddot{\alpha}_{i_2 \dots i_s]} - s \dot{b}_{i_2 \dots i_s]} \right) + \ddot{\beta}_{i_1 \dots i_s} + w^2 \beta_{i_1 \dots i_s} \\ w^2 b_{i_1 \dots i_{s-1}} - \frac{w^2}{s} \dot{\alpha}_{i_1 \dots i_s} \end{array} \right\} \\
\Box\omega &= \left\{ \begin{array}{c} ik_{[i_1} \left(\ddot{\alpha}_{i_2 \dots i_s]} + w^2 \alpha_{i_2 \dots i_s]} \right) + \ddot{\beta}_{i_1 \dots i_s} + w^2 \beta_{i_1 \dots i_s} \\ ik_{[i_1} \left(\ddot{a}_{i_2 \dots i_{s-1}]} + w^2 a_{i_2 \dots i_{s-1}]} \right) + \ddot{b}_{i_1 \dots i_{s-1}} + w^2 b_{i_1 \dots i_{s-1}} \end{array} \right\} .
\end{aligned}$$

We will refer to a form satisfying the wave equation $\Box\gamma = 0$ as “harmonic” even though this term usually refers to forms satisfying a Laplace equation. If γ satisfies $\Box\gamma = 0$ the objects $\alpha_{i_1 \dots i_{s-1}}$, $\beta_{i_1 \dots i_s}$, $a_{i_1 \dots i_{s-2}}$, and $b_{i_1 \dots i_{s-1}}$ can be parametrized as

$$\begin{aligned}
w\alpha_{i_1 \dots i_{s-1}}(\vec{k}, t) &= \alpha_{i_1 \dots i_{s-1}}(\vec{k})e^{-iwt} + \bar{\alpha}_{i_1 \dots i_{s-1}}(-\vec{k})e^{iwt} \\
\beta_{i_1 \dots i_s}(\vec{k}, t) &= \beta_{i_1 \dots i_s}(\vec{k})e^{-iwt} + \bar{\beta}_{i_1 \dots i_s}(-\vec{k})e^{iwt} \\
wa_{i_1 \dots i_{s-2}}(\vec{k}, t) &= a_{i_1 \dots i_{s-2}}(\vec{k})e^{-iwt} + \bar{a}_{i_1 \dots i_{s-2}}(-\vec{k})e^{iwt} \\
b_{i_1 \dots i_{s-1}}(\vec{k}, t) &= b_{i_1 \dots i_{s-1}}(\vec{k})e^{-iwt} + \bar{b}_{i_1 \dots i_{s-1}}(-\vec{k})e^{iwt}
\end{aligned} \tag{31}$$

For harmonic s -forms we have

$$d\gamma = \left\{ \begin{aligned} & ik_{[i_1} [\beta_{i_2 \dots i_{s+1}}](\vec{k}) e^{-iwt} + \bar{\beta}_{i_2 \dots i_{s+1}}(-\vec{k}) e^{iwt}] \\ & ik_{[i_1} \left[\frac{1}{s+1} \left(-i\alpha_{i_2 \dots i_s}(\vec{k}) - sb_{i_2 \dots i_s}(\vec{k}) \right) e^{-iwt} + \right. \\ & \quad \left. + \frac{1}{s+1} \left(i\bar{\alpha}_{i_2 \dots i_s}(-\vec{k}) - s\bar{b}_{i_2 \dots i_s}(-\vec{k}) \right) e^{iwt} \right] + \\ & \quad \left. - \frac{iw}{s+1} \beta_{i_1 \dots i_s}(\vec{k}) e^{-iwt} + \frac{iw}{s+1} \bar{\beta}_{i_1 \dots i_s}(-\vec{k}) e^{iwt} \right] \end{aligned} \right\}$$

$$\delta\gamma = \left\{ \begin{aligned} & ik_{[i_1} [-isa_{i_2 \dots i_{s-1}}](\vec{k}) e^{-iwt} + is\bar{a}_{i_2 \dots i_{s-1}}(-\vec{k}) e^{iwt}] + \\ & \quad + [w\alpha_{i_1 \dots i_{s-1}}(\vec{k}) - iwsb_{i_1 \dots i_{s-1}}(\vec{k})] e^{-iwt} + \\ & \quad + [w\bar{\alpha}_{i_1 \dots i_{s-1}}(-\vec{k}) + iws\bar{b}_{i_1 \dots i_{s-1}}(-\vec{k})] e^{iwt} \\ & - \frac{s}{s-1} wa_{i_1 \dots i_{s-2}}(\vec{k}) e^{-iwt} - \frac{s}{s-1} w\bar{a}_{i_1 \dots i_{s-2}}(-\vec{k}) e^{iwt} \end{aligned} \right\}$$

$$d\delta\gamma = \left\{ \begin{aligned} & ik_{[i_1} \left[\left(w\alpha_{i_2 \dots i_s}(\vec{k}) - iwsb_{i_2 \dots i_s}(\vec{k}) \right) e^{-iwt} + \right. \\ & \quad \left. + \left(w\bar{\alpha}_{i_2 \dots i_s}(-\vec{k}) + iws\bar{b}_{i_2 \dots i_s}(-\vec{k}) \right) e^{iwt} \right] \\ & - w^2 \left[b_{i_1 \dots i_{s-1}}(\vec{k}) + \frac{i}{s} \alpha_{i_1 \dots i_{s-1}}(\vec{k}) \right] e^{-iwt} - \\ & \quad - w^2 \left[\bar{b}_{i_1 \dots i_{s-1}}(-\vec{k}) - \frac{i}{s} \bar{\alpha}_{i_1 \dots i_{s-1}}(-\vec{k}) \right] e^{iwt} \end{aligned} \right\}$$

$$\delta d\gamma = \left\{ \begin{aligned} & ik_{[i_1} \left[- \left(w\alpha_{i_2 \dots i_s}(\vec{k}) - iwsb_{i_2 \dots i_s}(\vec{k}) \right) e^{-iwt} - \right. \\ & \quad \left. - \left(w\bar{\alpha}_{i_2 \dots i_s}(-\vec{k}) + iws\bar{b}_{i_2 \dots i_s}(-\vec{k}) \right) e^{iwt} \right] \\ & w^2 \left[b_{i_1 \dots i_{s-1}}(\vec{k}) + \frac{i}{s} \alpha_{i_1 \dots i_{s-1}}(\vec{k}) \right] e^{-iwt} + \\ & \quad + w^2 \left[\bar{b}_{i_1 \dots i_{s-1}}(-\vec{k}) - \frac{i}{s} \bar{\alpha}_{i_1 \dots i_{s-1}}(-\vec{k}) \right] e^{iwt} \end{aligned} \right\}$$

and trivially

$$\square\gamma = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}.$$

Appendix B. Solutions to $d\Box\omega = 0$ and $\delta\Box\omega = 0$.

The general solution to the system of equations

$$d\Box\omega = 0$$

$$\delta\Box\omega = 0$$

can be written as $\omega = \gamma + d\delta\gamma_{HD}$ where γ is a general solution to $\Box\omega = 0$ and γ_{HD} is a general solution to $\Box^2\omega = 0$. We prove in the following that such general solution can be parametrized as

$$\gamma = \left\{ \begin{array}{l} \beta_{i_1 \dots i_s}(\vec{k})e^{-iwt} + \bar{\beta}_{i_1 \dots i_s}(-\vec{k})e^{iwt} \\ \frac{i}{w}k_{[i_1} \left[a_{i_2 \dots i_{s-1}}(\vec{k})e^{-iwt} + \bar{a}_{i_2 \dots i_{s-1}}(-\vec{k})e^{iwt} \right] + \\ + b_{i_1 \dots i_{s-1}}(\vec{k})e^{-iwt} + \bar{b}_{i_1 \dots i_{s-1}}(-\vec{k})e^{iwt} \end{array} \right\}, \quad (32)$$

$$d\delta\gamma_{HD} = \left\{ \begin{array}{l} \frac{i}{w}k_{[i_1} \left\{ \left[\alpha_{i_2 \dots i_s}(\vec{k}) + wt\sigma_{i_2 \dots i_s}(\vec{k}) \right] e^{-iwt} + \right. \\ \left. + \left[\bar{\alpha}_{i_2 \dots i_s}(-\vec{k}) + wt\bar{\sigma}_{i_2 \dots i_s}(-\vec{k}) \right] e^{iwt} \right\} \\ \frac{1}{s} \left[\sigma_{i_1 \dots i_{s-1}}(\vec{k}) - i\alpha_{i_1 \dots i_{s-1}}(\vec{k}) - iwt\sigma_{i_1 \dots i_{s-1}}(\vec{k}) \right] e^{-iwt} + \\ + \frac{1}{s} \left[\bar{\sigma}_{i_1 \dots i_{s-1}}(-\vec{k}) + i\bar{\alpha}_{i_1 \dots i_{s-1}}(-\vec{k}) + iwt\bar{\sigma}_{i_1 \dots i_{s-1}}(-\vec{k}) \right] e^{iwt} \end{array} \right\} \quad (33)$$

To this end it suffices to write a general solution to $\Box^2\omega = 0$, compute $d\delta\omega$, add to it a general solution to the wave equation, and absorb in a single term those that may be written as solutions to either equation. A general harmonic form is given by (31) and a general solution to $\Box^2\omega = 0$ can be parametrized as

$$\gamma_{HD} = \left\{ \begin{array}{l} ik_{[i_1} \left\{ \left[\rho_{i_2 \dots i_s}(\vec{k}) + \frac{t}{w^2}\sigma_{i_2 \dots i_s}(\vec{k}) \right] e^{iwt} + \right. \\ \left. + \left[\bar{\rho}_{i_2 \dots i_s}(-\vec{k}) + \frac{t}{w^2}\bar{\sigma}_{i_2 \dots i_s}(-\vec{k}) \right] e^{-iwt} \right\} \\ + \left[\mu_{i_1 \dots i_s}(\vec{k}) + wt\nu_{i_1 \dots i_s}(\vec{k}) \right] e^{iwt} + \left[\bar{\mu}_{i_1 \dots i_s}(-\vec{k}) + wt\bar{\nu}_{i_1 \dots i_s}(-\vec{k}) \right] e^{-iwt} \\ ik_{[i_1} \left\{ \left[r_{i_2 \dots i_{s-1}}(\vec{k}) + wts_{i_2 \dots i_{s-1}}(\vec{k}) \right] e^{iwt} + \right. \\ \left. + \left[\bar{r}_{i_2 \dots i_{s-1}}(-\vec{k}) + wt\bar{s}_{i_2 \dots i_{s-1}}(-\vec{k}) \right] e^{-iwt} \right\} \\ + \left[m_{i_1 \dots i_{s-1}}(\vec{k}) + wtn_{i_1 \dots i_{s-1}}(\vec{k}) \right] e^{iwt} + \\ + \left[\bar{m}_{i_1 \dots i_{s-1}}(-\vec{k}) + wt\bar{n}_{i_1 \dots i_{s-1}}(-\vec{k}) \right] e^{-iwt} \end{array} \right\}$$

and $d\delta$ acting on a general solution to $\square^2\omega = 0$ is given by

$$\left\{ \begin{aligned} & ik_{[i_1} \left\{ \left[sw n_{i_2 \dots i_s}(\vec{k}) - isw m_{i_2 \dots i_s}(\vec{k}) + w^2 \rho_{i_2 \dots i_s}(\vec{k}) - \right. \right. \\ & \quad \left. \left. - isw^2 t n_{i_2 \dots i_s}(\vec{k}) + t \sigma_{i_2 \dots i_s}(\vec{k}) \right] e^{-iwt} + \right. \\ & \quad \left. + \left[sw \bar{n}_{i_2 \dots i_s}(-\vec{k}) + isw \bar{m}_{i_2 \dots i_s}(-\vec{k}) + w^2 \bar{\rho}_{i_2 \dots i_s}(-\vec{k}) + \right. \right. \\ & \quad \left. \left. + isw^2 t \bar{n}_{i_2 \dots i_s}(-\vec{k}) + t \bar{\sigma}_{i_2 \dots i_s}(-\vec{k}) \right] e^{iwt} \right\} \\ & ik_{[i_1} \left[-2iw^2 s_{i_2 \dots i_{s-1}}(\vec{k}) e^{-iwt} + 2iw^2 \bar{s}_{i_2 \dots i_{s-1}}(-\vec{k}) e^{iwt} \right] - \\ & \quad - \left[2iw^2 n_{i_1 \dots i_{s-1}}(\vec{k}) + w^2 m_{i_1 \dots i_{s-1}}(\vec{k}) - \right. \\ & \quad \left. - \frac{1}{s} \sigma_{i_1 \dots i_{s-1}}(\vec{k}) + \frac{iw^3}{s} \rho_{i_1 \dots i_{s-1}}(\vec{k}) + \right. \\ & \quad \left. + w^3 t n_{i_1 \dots i_{s-1}}(\vec{k}) + \frac{iwt}{s} \sigma_{i_1 \dots i_{s-1}}(\vec{k}) \right] e^{-iwt} + \\ & \quad + \left[2iw^2 \bar{n}_{i_1 \dots i_{s-1}}(-\vec{k}) - w^2 \bar{m}_{i_1 \dots i_{s-1}}(-\vec{k}) + \right. \\ & \quad \left. + \frac{1}{s} \bar{\sigma}_{i_1 \dots i_{s-1}}(-\vec{k}) + \frac{iw^3}{s} \bar{\rho}_{i_1 \dots i_{s-1}}(-\vec{k}) - \right. \\ & \quad \left. - w^3 t \bar{n}_{i_1 \dots i_{s-1}}(-\vec{k}) + \frac{iwt}{s} \bar{\sigma}_{i_1 \dots i_{s-1}}(-\vec{k}) \right] e^{iwt} \end{aligned} \right\}.$$

Looking at the previous two expressions we see that the terms involving time dependent exponentials and objects such as wte^{iwt} appear in the right combinations to allow the field redefinitions leading us to write (32) and (33).

Appendix C. The algebraic constraints.

The field equations are

$$P\delta dA + Qd\delta A = 0 \quad . \quad (34)$$

The general solution to it has been obtained by solving a set of necessary conditions and substituting them back into (34). In this way we get

$$P(\delta d\gamma^q e_q + \delta d\gamma^r e_r) + Q(d\delta\gamma^p e_p + d\delta\gamma^r e_r + d\delta d\delta\gamma_{HD}^r e_r) = 0 \quad .$$

Taking into account that

$$\delta d\gamma^q = \left\{ \begin{aligned} & ik_{[i_1} \left(iws b_{i_2 \dots i_s}^q(\vec{k}) e^{-iwt} - iws \bar{b}_{i_2 \dots i_s}^q(-\vec{k}) e^{iwt} \right) \\ & w^2 b_{i_1 \dots i_{s-1}}^q(\vec{k}) e^{-iwt} + w^2 \bar{b}_{i_1 \dots i_{s-1}}^q(-\vec{k}) e^{iwt} \end{aligned} \right\}$$

$$\begin{aligned}
\delta d\gamma^r &= \left\{ \begin{aligned} &ik_{[i_1} \left(iwsb_{i_2 \dots i_s]}^r(\vec{k})e^{-iwt} - iws\bar{b}_{i_2 \dots i_s]}^r(-\vec{k})e^{iwt} \right) \\ &w^2b_{i_1 \dots i_{s-1}}^r(\vec{k})e^{-iwt} + w^2\bar{b}_{i_1 \dots i_{s-1}}^r(-\vec{k})e^{iwt} \end{aligned} \right\} \\
d\delta\gamma^p &= \left\{ \begin{aligned} &ik_{[i_1} \left(w\alpha_{i_2 \dots i_s]}^p(\vec{k})e^{-iwt} + w\bar{\alpha}_{i_2 \dots i_s]}^p(-\vec{k})e^{iwt} \right) \\ &-\frac{iw^2}{s}\alpha_{i_1 \dots i_{s-1}}^p(\vec{k})e^{-iwt} + \frac{iw^2}{s}\bar{\alpha}_{i_1 \dots i_{s-1}}^p(-\vec{k})e^{iwt} \end{aligned} \right\} \\
d\delta\gamma^r &= \left\{ \begin{aligned} &ik_{[i_1} \left(-iswb_{i_2 \dots i_s]}^r(\vec{k})e^{-iwt} + isw\bar{b}_{i_2 \dots i_s]}^r(-\vec{k})e^{iwt} \right) \\ &-w^2b_{i_1 \dots i_{s-1}}^r(\vec{k})e^{-iwt} - w^2\bar{b}_{i_1 \dots i_{s-1}}^r(-\vec{k})e^{iwt} \end{aligned} \right\} \\
d\delta d\gamma_{HD}^r &= \left\{ \begin{aligned} &ik_{[i_1} \left(-2iws\sigma_{i_2 \dots i_s]}^r(\vec{k})e^{-iwt} + 2iws\bar{\sigma}_{i_2 \dots i_s]}^r(-\vec{k})e^{iwt} \right) \\ &-\frac{2w^2}{s}\sigma_{i_1 \dots i_{s-1}}^r(\vec{k})e^{-iwt} - \frac{2w^2}{s}\bar{\sigma}_{i_1 \dots i_{s-1}}^r(-\vec{k})e^{iwt} \end{aligned} \right\} ,
\end{aligned}$$

we find the Fourier transform of the spatial part of the constraint

$$\begin{aligned}
&P \left[iswb_{i_1 \dots i_{s-1}}^q(\vec{k})e_q + iswb_{i_1 \dots i_{s-1}}^r(\vec{k})e_r \right] + \\
&+Q \left[w\alpha_{i_1 \dots i_{s-1}}^p(\vec{k})e_p - iswb_{i_1 \dots i_{s-1}}^r(\vec{k})e_r - 2iws\sigma_{i_1 \dots i_{s-1}}^r(\vec{k})e_r \right] = 0
\end{aligned}$$

and its time part

$$\begin{aligned}
&P \left[w^2b_{i_1 \dots i_{s-1}}^q(\vec{k})e_q + w^2b_{i_1 \dots i_{s-1}}^r(\vec{k})e_r \right] + \\
&+Q \left[-iw^2s^{-1}\alpha_{i_1 \dots i_{s-1}}^p(\vec{k})e_p - w^2b_{i_1 \dots i_{s-1}}^r(\vec{k})e_r - 2w^2s^{-1}\sigma_{i_1 \dots i_{s-1}}^r(\vec{k})e_r \right] = 0 ,
\end{aligned}$$

which can be collected in the algebraic constraint

$$\begin{aligned}
&P \left[b_{i_1 \dots i_{s-1}}^q(\vec{k})e_q + b_{i_1 \dots i_{s-1}}^r(\vec{k})e_r \right] = \\
&= Q \left[\frac{i}{s}\alpha_{i_1 \dots i_{s-1}}^p(\vec{k}) + b_{i_1 \dots i_{s-1}}^r(\vec{k})e_r + \frac{2}{s}\sigma_{i_1 \dots i_{s-1}}^r(\vec{k})e_r \right] .
\end{aligned}$$

Appendix D.

In order to compute the symplectic structure we need

$$\begin{aligned}
\gamma^q &= \left\{ \begin{array}{l} \beta_{i_1 \dots i_s}^q(\vec{k})e^{-iwt} + \bar{\beta}_{i_1 \dots i_s}^q(-\vec{k})e^{iwt} \\ b_{i_1 \dots i_{s-1}}^q(\vec{k})e^{-iwt} + \bar{b}_{i_1 \dots i_{s-1}}^q(-\vec{k})e^{iwt} \end{array} \right\} \\
\gamma^p &= \left\{ \begin{array}{l} \frac{i}{w}k_{[i_1} [\alpha_{i_2 \dots i_s}^p(\vec{k})e^{-iwt} + \bar{\alpha}_{i_2 \dots i_s}^p(-\vec{k})e^{iwt}] \\ \frac{i}{w}k_{[i_1} [a_{i_2 \dots i_{s-1}}^p(\vec{k})e^{-iwt} + \bar{a}_{i_2 \dots i_{s-1}}^p(-\vec{k})e^{iwt}] \end{array} \right\} \\
\gamma^r &= \left\{ \begin{array}{l} \beta_{i_1 \dots i_s}^r(\vec{k})e^{-iwt} + \bar{\beta}_{i_1 \dots i_s}^r(-\vec{k})e^{iwt} \\ \frac{i}{w}k_{[i_1} [a_{i_2 \dots i_{s-1}}^r(\vec{k})e^{-iwt} + \bar{a}_{i_2 \dots i_{s-1}}^r(-\vec{k})e^{iwt}] + \\ \quad + b_{i_1 \dots i_{s-1}}^r(\vec{k})e^{-iwt} + \bar{b}_{i_1 \dots i_{s-1}}^r(-\vec{k})e^{iwt} \end{array} \right\} \\
d\delta\gamma_{HD}^r &= \left\{ \begin{array}{l} \frac{i}{w}k_{[i_1} \left\{ [\alpha_{i_2 \dots i_s}^r(\vec{k}) + wt\sigma_{i_2 \dots i_s}^r(\vec{k})]e^{-iwt} + \right. \\ \quad \left. + [\bar{\alpha}_{i_2 \dots i_s}^r(-\vec{k}) + wt\bar{\sigma}_{i_2 \dots i_s}^r(-\vec{k})]e^{iwt} \right\} \\ \frac{1}{s} \left[(\sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) - i\alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) - iwt\sigma_{i_1 \dots i_{s-1}}^r(\vec{k}))e^{-iwt} + \right. \\ \quad \left. + (\bar{\sigma}_{i_1 \dots i_{s-1}}^r(-\vec{k}) + i\bar{\alpha}_{i_1 \dots i_{s-1}}^r(-\vec{k}) + iwt\bar{\sigma}_{i_1 \dots i_{s-1}}^r(-\vec{k}))e^{iwt} \right] \end{array} \right\} \\
d\gamma^q &= \left\{ \begin{array}{l} ik_{[i_1} [\beta_{i_2 \dots i_{s+1}}^q(\vec{k})e^{-iwt} + \bar{\beta}_{i_2 \dots i_{s+1}}^q(-\vec{k})e^{iwt}] \\ ik_{[i_1} \left[-\frac{s}{s+1}b_{i_2 \dots i_s}^q(\vec{k})e^{-iwt} - \frac{s}{s+1}\bar{b}_{i_2 \dots i_s}^q(-\vec{k})e^{iwt} \right] - \\ \quad - \frac{iw}{s+1}\beta_{i_1 \dots i_s}^q(\vec{k})e^{-iwt} + \frac{iw}{s+1}\bar{\beta}_{i_1 \dots i_s}^q(-\vec{k})e^{iwt} \end{array} \right\} \\
d\gamma^r &= \left\{ \begin{array}{l} ik_{[i_1} [\beta_{i_2 \dots i_{s+1}}^r(\vec{k})e^{-iwt} + \bar{\beta}_{i_2 \dots i_{s+1}}^r(-\vec{k})e^{iwt}] \\ ik_{[i_1} \left[-\frac{s}{s+1}b_{i_2 \dots i_s}^r(\vec{k})e^{-iwt} - \frac{s}{s+1}\bar{b}_{i_2 \dots i_s}^r(-\vec{k})e^{iwt} \right] - \\ \quad - \frac{iw}{s+1}\beta_{i_1 \dots i_s}^r(\vec{k})e^{-iwt} + \frac{iw}{s+1}\bar{\beta}_{i_1 \dots i_s}^r(-\vec{k})e^{iwt} \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
\delta\gamma^p &= \left\{ \begin{aligned} &ik_{[i_1} \left[-isa_{i_1 \dots i_{s-1}}^p(\vec{k})e^{-iwt} + is\bar{a}_{i_1 \dots i_{s-1}}^p(-\vec{k})e^{iwt} \right] + \\ &\quad + w\alpha_{i_1 \dots i_{s-1}}^p(\vec{k})e^{-iwt} + w\bar{\alpha}_{i_1 \dots i_{s-1}}^p(-\vec{k})e^{iwt} \\ &-\frac{s}{s-1}wa_{i_1 \dots i_{s-2}}^p(\vec{k})e^{-iwt} - \frac{s}{s-1}w\bar{a}_{i_1 \dots i_{s-2}}^p(-\vec{k})e^{iwt} \end{aligned} \right\} \\
\delta\gamma^r &= \left\{ \begin{aligned} &ik_{[i_1} \left[-isa_{i_1 \dots i_{s-1}}^r(\vec{k})e^{-iwt} + is\bar{a}_{i_1 \dots i_{s-1}}^r(-\vec{k})e^{iwt} \right] - \\ &\quad - iswb_{i_1 \dots i_{s-1}}^r(\vec{k})e^{-iwt} + isw\bar{b}_{i_1 \dots i_{s-1}}^r(-\vec{k})e^{iwt} \\ &-\frac{s}{s-1}wa_{i_1 \dots i_{s-2}}^r(\vec{k})e^{-iwt} - \frac{s}{s-1}w\bar{a}_{i_1 \dots i_{s-2}}^r(-\vec{k})e^{iwt} \end{aligned} \right\} \\
\delta d\delta\gamma_{HD}^r &= \left\{ \begin{aligned} &-2iw\sigma_{i_1 \dots i_{s-1}}^r(\vec{k})e^{-iwt} + 2iw\bar{\sigma}_{i_1 \dots i_{s-1}}^r(-\vec{k})e^{iwt} \\ &0 \end{aligned} \right\}.
\end{aligned}$$

The symplectic form associated to the action (3) with $\mathcal{M} = \mathbb{R}^4$ is

$$\Omega_{\mathcal{F}} = \int_{\mathbb{R}^3} \left[d\mathbb{A}^{\mathfrak{t}} \mathbb{A} * P d\mathbb{A} + (Q\delta d\mathbb{A})^{\mathfrak{t}} \mathbb{A} * d\mathbb{A} \right]. \quad (35)$$

We must compute the restriction of (35) to the space of solutions to the field equations

\mathcal{S} . This gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[d\mathbb{A}^{\mathfrak{t}} \mathbb{A} * P d\mathbb{A} + (Q\delta d\mathbb{A})^{\mathfrak{t}} \mathbb{A} * d\mathbb{A} \right] \Big|_{\mathcal{S}} = \\
&= \int_{\mathbb{R}^3} (d\mathbb{I}\gamma^q e_q^{\mathfrak{t}} + d\mathbb{I}\gamma^r e_r^{\mathfrak{t}}) \mathbb{A} * P(d\mathbb{I}\gamma^{q'} e_{q'} + d\mathbb{I}\gamma^{r'} e_{r'}) + \\
&+ \int_{\mathbb{R}^3} d\mathbb{I}\Lambda^q e_q^{\mathfrak{t}} \mathbb{A} * P(d\mathbb{I}\gamma^{q'} e_{q'} + d\mathbb{I}\gamma^{r'} e_{r'}) + \\
&+ \int_{\mathbb{R}^3} (Q(\delta d\mathbb{I}\gamma^p e_p + (\delta d\mathbb{I}\gamma^r + \delta d\delta d\mathbb{I}\gamma_{HD}^r) e_r))^{\mathfrak{t}} \mathbb{A} * \delta d\Theta^p e_p + \\
&+ \int_{\mathbb{R}^3} (Q(\delta d\mathbb{I}\gamma^p e_p + \delta d\mathbb{I}\gamma^r e_r))^{\mathfrak{t}} \mathbb{A} * (d\mathbb{I}\gamma^{p'} e_{p'} + d\mathbb{I}\gamma^{r'} e_{r'}) + \\
&+ \int_{\mathbb{R}^3} (Q\delta d\delta d\mathbb{I}\gamma_{HD}^r e_r)^{\mathfrak{t}} \mathbb{A} * (d\mathbb{I}\gamma^{p'} e_{p'} + d\mathbb{I}\gamma^{r'} e_{r'}) + \\
&+ \int_{\mathbb{R}^3} (Q(\delta d\mathbb{I}\gamma^p e_p + (\delta d\mathbb{I}\gamma^r + \delta d\delta d\mathbb{I}\gamma_{HD}^r) e_r))^{\mathfrak{t}} \mathbb{A} * d\delta d\mathbb{I}\gamma_{HD}^{r'} e_{r'} +
\end{aligned} \quad (36)$$

$$+ \int_{\mathbf{R}^3} d\delta \mathbb{d}\gamma_{HD}^r e_r^{\mathbf{t}} \mathbb{A} * P(d\mathbb{d}\gamma^{q'} e_{q'} + d\mathbb{d}\gamma^{r'} e_{r'}) \quad .$$

To proceed further¹⁹ we need the following formula for a s -form $\overset{(1)}{\omega}$ and a $(s+1)$ -form $\overset{(2)}{\omega}$

$$\begin{aligned} \int_{\mathbf{R}^n} \overset{(1)}{\omega} \wedge * \overset{(2)}{\omega} &= (s+1)! \int_{\mathbf{R}^n} \frac{d^n \vec{k}}{w^2} \beta_{i_1 \dots i_s}^{(1)}(\vec{k}, t) b_{i_1 \dots i_s}^{(2)}(-\vec{k}, t) \\ &+ \frac{(s+1)!}{s} \int_{\mathbf{R}^n} d^n \vec{k} \alpha_{i_1 \dots i_{s-1}}^{(1)}(\vec{k}, t) a_{i_1 \dots i_{s-1}}^{(2)}(-\vec{k}, t) \quad . \end{aligned}$$

The first term in (36) can be easily computed just by substituting the solutions to the field equations and taking into account that integrals such as

$$\int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \mathbb{d}\beta_{i_1 \dots i_s}^{\mathbf{t}}(\vec{k}) \mathbb{A} P \mathbb{d}\beta_{i_1 \dots i_s}(-\vec{k}) = 0 \quad ,$$

due to the fact that P is symmetric and $\mathbb{d}\beta$ in a 1-form in the solution space. This way we get

$$\int_{\mathbf{R}^3} \mathbb{d}\gamma^{\mathbf{t}} \mathbb{A} * P d\mathbb{d}\gamma = -2is! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \mathbb{d}\bar{\beta}_{i_1 \dots i_s}^{\mathbf{t}}(\vec{k}) \mathbb{A} P \mathbb{d}\beta_{i_1 \dots i_s}(\vec{k}) \quad .$$

The second term in (36) can be easily seen to be zero as a consequence of the algebraic constraint and the fact that $Qe_q = 0$. The third term is also zero because

$$Q[\delta \mathbb{d}\gamma^p e_p + (\delta \mathbb{d}\gamma^r + \delta d\delta \mathbb{d}\gamma_{HD}^r) e_r]$$

can be written as the matrix P acting on something and $Pe_p = 0$. The computation of the sum of the fourth and fifth term is straightforward and gives

$$\begin{aligned} \int_{\mathbf{R}^3} (Q(\delta \mathbb{d}\gamma^p e_p + \delta \mathbb{d}\gamma^r e_r))^{\mathbf{t}} \mathbb{A} * (\mathbb{d}\gamma^{p'} e_{p'} + \mathbb{d}\gamma^{r'} e_{r'}) + \\ + \int_{\mathbf{R}^3} (Q\delta d\delta \mathbb{d}\gamma_{HD}^r e_r)^{\mathbf{t}} \mathbb{A} * (\mathbb{d}\gamma^{p'} e_{p'} + \mathbb{d}\gamma^{r'} e_{r'}) = \end{aligned}$$

¹⁹Notice that d and δ are four dimensional operators but the integrals extend only to \mathbf{R}^3 so that we cannot integrate by parts.

$$\begin{aligned}
&= \frac{2iss!}{s-1} \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \mathbb{d} \bar{a}_{i_1 \dots i_{s-2}}^{\mathbf{t}}(\vec{k}) \mathbb{A} Q \mathbb{d} a_{i_1 \dots i_{s-2}}(\vec{k}) + \\
&\quad + 2iss! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \mathbb{d} \bar{b}_{i_1 \dots i_{s-1}}^r e_r^{\mathbf{t}}(\vec{k}) \mathbb{A} P \mathbb{d} b_{i_1 \dots i_{s-1}}(\vec{k}) \quad .
\end{aligned}$$

Finally the last two terms require a somewhat lengthy computation that is simplified by the following remarks:

First of all, the terms proportional to e^{2iwt} and e^{-2iwt} are both zero. Second

$$\begin{aligned}
&\int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \left\{ \mathbb{d} \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \left[\mathbb{d} \bar{b}_{i_1 \dots i_{s-1}}^q(\vec{k}) e_q + \mathbb{d} \bar{b}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right] + \right. \\
&\quad \left. + \mathbb{d} \bar{\sigma}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \left[\mathbb{d} b_{i_1 \dots i_{s-1}}^q(\vec{k}) e_q + \mathbb{d} b_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right] \right\} = 0
\end{aligned}$$

as can be seen by using the algebraic constraint to write

$$\begin{aligned}
P \left[\mathbb{d} b_{i_1 \dots i_{s-1}}^q(\vec{k}) e_q + \mathbb{d} b_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right] &= \\
&= Q \left[\frac{i}{s} \mathbb{d} \alpha_{i_1 \dots i_{s-1}}^p(\vec{k}) + \mathbb{d} b_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r + \frac{2}{s} \mathbb{d} \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r \right] \quad ,
\end{aligned}$$

and the fact that $\mathbb{d} A \mathbb{A} \mathbb{d} B = -\mathbb{d} B \mathbb{A} \mathbb{d} A$ for 1-forms.

So we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^3} d\delta \mathbb{d} \gamma_{HD}^r e_r^{\mathbf{t}} \mathbb{A} * P(d\mathbb{d} \gamma^{q'} e_{q'} + d\mathbb{d} \gamma^{r'} e_{r'}) + \\
&\quad + \int_{\mathbf{R}^3} (Q(\delta \mathbb{d} \gamma^p e_p + (\delta \mathbb{d} \gamma^r + \delta d\delta \mathbb{d} \gamma_{HD}^r e_r)) e_r^{\mathbf{t}} \mathbb{A} * d\delta \mathbb{d} \gamma_{HD}^{r'} e_{r'} = \\
&= is! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \left[\mathbb{d} \bar{\sigma}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d} b_{i_1 \dots i_{s-1}}(\vec{k}) - \mathbb{d} \sigma_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d} \bar{b}_{i_1 \dots i_{s-1}}(\vec{k}) \right] \\
&\quad - 2s! \int_{\mathbf{R}^3} \frac{d^3 \vec{k}}{w} \left[\mathbb{d} \bar{\alpha}_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d} b_{i_1 \dots i_{s-1}}(\vec{k}) + \mathbb{d} \alpha_{i_1 \dots i_{s-1}}^r(\vec{k}) e_r^{\mathbf{t}} \mathbb{A} P \mathbb{d} \bar{b}_{i_1 \dots i_{s-1}}(\vec{k}) \right] .
\end{aligned}$$

Adding up all these contributions we finally get (22).

Appendix E.

If a diagonal element in a quadratic form \mathcal{Q} is zero, and the row and column where this element is are not identically zero, then \mathcal{Q} cannot be either definite or semidefinite.

The proof is very simple; let us write

$$\mathcal{Q} = \begin{bmatrix} 0 & v^{\mathfrak{t}} \\ v & q \end{bmatrix}$$

then

$$X^{\mathfrak{t}} \mathcal{Q} X = \begin{bmatrix} x^{\mathfrak{t}} & y^{\mathfrak{t}} \end{bmatrix} \begin{bmatrix} 0 & v^{\mathfrak{t}} \\ v & q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2y^{\mathfrak{t}}vx + y^{\mathfrak{t}}qy. \quad (37)$$

Let us fix y such that $y^{\mathfrak{t}}v \neq 0$. We have then that (37) can take both positive and negative values depending on the choice of x .

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